International Journal of Theoretical Physics, Vol. 33, No. 10, 1994

Parametrization of the Cosmological Scale Factor

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Received April 11, 1994

We show that the growth of the cosmological scale factor $R(\lambda)$ can be conveniently parametrized as a function of a space-time-dependent cosmological function $\lambda(x)$. To show the parametrization, we introduce a simple heuristic model of the cosmological function $\lambda(x)$ during the inflationary period by assuming that it is spatially uniform but time dependent with an exponential growth phase followed by a rapid decay. Based upon this relatively simple empirical model we are able to calculate directly all the required features of an inflationary period such as exponential growth of the scale factor plus a natural relaxation (graceful exit) of λ to a *positive* present-day cosmological constant. The model also predicts the presence of a very large, negative Planckian cosmological constant at the Planck time.

1. INTRODUCTION

The understanding of the present-day cosmological constant λ_0 is at best experimentally incomplete. Observational data on remote quasistellar objects do not fit simple cosmological models without a cosmological constant (Zeldovitch, 1968, 1981). Also the introduction of $\lambda \neq 0$ signifies that the vacuum energy-momentum, i.e., empty space, produces a gravitational field just as well as matter energy-momentum. That the vacuum can contribute to the energy density of spacetime is not surprising in view of modern field theories of elementary particles. Modern field theories not only allow for a nonzero vacuum energy density, but they also strongly suggest it should have a large value (Georgi and Glashow, 1974). For example, in most inflationary models (Albrecht *et al.*, 1982; Suen and Clifford, 1988; Abbott, 1988); the vacuum energy density may well reach $10^{18}\,\mathrm{GeV}^2$. Thus it is most puzzling for cosmological models that the

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0020-7748/94/1000-2099$07.00/0

cosmological constant now takes the extraordinary small value (Hawking, 1984), $|\lambda_0| \leqq 10^{-82}\ \text{GeV}^2$. It is very difficult to explain in any inflationary model how a factor of 10^{100} could occur in any "natural" way. It is not just simply a case of repeating the oft-quoted statement, "All that is needed for inflation is for the cosmological constant to be sufficiently large for a sufficiently long period of time for the universe to expand by at least 30 orders of magnitude." First of all the cosmological "constant" can not be constant, and second its change must be related to some physical process that feeds into the gravitational field equations which govern the time evolution of the universe. Since the time dependence of the cosmological function is relevant to its present-day value, it seems ironic that this *relic* of the inflationary period has not been treated more carefully in inflationary models. This problem is probably the most serious deficiency of all inflationary models. Thus the important question occurs within these models: Is it possible to model the inflationary period with a well-defined cosmological *function* that eventually decreases to the present-day value? This, of course, leaves open the fate of the cosmological function *after* the end of the inflationary period. The heuristic model used here will imply a very different possibility, which we touch on when appropriate and in our conclusions.

In order to study some aspects of the above problems, we first model the inflationary period. We present an ansatz for the cosmological function which can be related to the time of inflation that depends on a period of exponential growth followed by exponential decay. Such a model shows that the cosmological function must start out at the Planck time with an extraordinarily large value in comparison with its present-day value in order to cancel what appears also to be a very large, negative Planckian cosmological constant. Physically, a Planckian cosmological constant signifies that the vacuum must be extraordinarily stable so that the initial instability must have been very large to lead to an inflationary epoch. The time dependence of the cosmological function then provides a rate scale for events during the inflationary period. Thus at the beginning of inflation when the temperature T is of the order of the Planck mass ($\simeq 10^{19}$ GeV), the total energy density of the universe is dominated by the vacuum energy density. Due to an instability of the vacuum energy density, there will be a transition to an inflationary period. The cosmological function then acts as an effective energy-momentum tensor in the gravitational field equations from which we are able to predict the growth of the *cosmological scale factor*. Our heuristic model does not strongly depend on the details of the inflationary period; however, the physics that manifests itself in the cosmological function which in turn produces an effective energy-momentum tensor is as yet unknown or unproven. Thus referring to this process in terms of the growth and decay of an "X particle" could be misleading and will be avoided for the most part. However, the details are not inconsistent with different theories such as the grand unified theory (GUT), quantum mechanical models, spacetime symmetry breaking, or chaotic models, since the overriding similarity among these models is the time scales. The major events are closely paralleled by the growth and decay of the cosmological function, which in turn shows the significance of the parametrization we discuss in the body of this work. In the next section, however, we discuss the motivations for a time-varying cosmological function, and in Section 3 we present a heuristic model for $\lambda(t)$. In this work the cosmological function is constrained to be spatially uniform for simplicity; such variations in themselves could have significance, but will not be considered here.

In Section 4 we represent the field equations and the energy-momentum tensor. In Section 5, we prove that there exists a consistency among the field equations and the conservation of energy-momentum that depends on the cosmological function, the model of which we present in Section 6 as well as how we set the parameters using boundary conditions. We then give our calculation of the scale factor in Section 7, and present our conclusions and recommendations in Section 8.

2. MOTIVATION: EXISTENCE OF THE COSMOLOGICAL CONSTANT

2.1. Future Observational Tests

Up to the present, the most positive recent discussion of observational evidence for a nonzero cosmological term has involved distance/time-scale arguments by de Vaucouleurs (1982, 1983*a*,*b*) and space distributions of the quasars by Fliche and Souriax (1992). A significant negative discussion, based on the redshift-number test, was presented by Loh (1986). Loh's discussion was, however, critically reviewed by Weinberg (1982) and the issue of the cosmological constant remains very much open. Turner *et al.* (1984) have reached a similar conclusion based on a comprehensive survey of all the available observational evidence, all given equal weight. Klapdor and Grotz (1986) have found similar evidence in favor of a nonvanishing cosmological term from knowledge of better cross sections for heavy-elements synthesis and further investigation of globular clusters, which gave revised estimates based on beta-delayed fission and neutron emission studies of the *r*-process by Thielemann *et al.* (1983). Similar conclusions were presented by Blome and Priester (1985) along with a subsequent analysis of the role of the vacuum energy in cosmology.

In subsequent analyses of potential observations with better telescopes, such as the Hubble Space Telescope, Hoell Chu *et al.* (1988) have focused on the apparent diameter vs. redshift test for galaxies to provide sufficient information for a reliable estimate of the cosmological term. Careful reading of a recent paper by Loh (1988) leads to similar conclusions concerning the desirability of a series of accurate measurements with an instrument of the capability of the Hubble Space Telescope. In anticipation of this, a complete set of astrophysical formulas for all the standard tests in cosmology has been made available in the work of Dabrowski and Stelmach (1982, 1986). The Hubble Space Telescope will therefore be of great value in providing observations which should provide better upper and lower limits on the cosmological term and other cosmological parameters.

2.2. Asymptotic and Qualitative Analyses of Cosmological Models with a Cosmological Term

Qualitative and geometric analyses of cosmological models with a cosmological term have been limited in number. An early work by Nikomarov and Khalatnikov (1978) considered only Friedmann models and was related to prior work on dissipative processes and bulk viscosity in those models by Belinskii and Khalatnikov (1976, 1977). More extensive work by Weber (1984, 1985, 1987, 1988) has considered the role of a cosmological constant in the evolution of an anisotropic universe from early through late times. This work has still been somewhat limited in that inhomogeneous cosmologies have not been analyzed, nor have all the Bianchi types. Asymptotic analyses of the late-time effects of a cosmological constant have been given by Fabbri (1979) and Wald (1983). Those asymptotic analyses have been extended to further establish a correspondence between the late-time behaviors of inflationary cosmologies and those with cosmological constant by Jensen and Stein-Schabes (1986, 1987) and by Turner and Widrow (1986).

None of these analyses have considered the entire range of Bianchi types, nor have they approached the question of the analysis of such models with a *time-varying* cosmological term. A direction for future work which would be most useful is the qualitative and asymptotic analyses of the general classes of spatially homogeneous cosmologies with various time-varying cosmological terms, to obtain limits and constraints of the functional forms of those terms (as functions of time).

2.3. Analyses of Other Approaches to Time-Varying Cosmological Terms and Inflationary Cosmologies

A large number of independent investigations of various effects of time-varying cosmological terms on the evolution of cosmological models have appeared. It would be useful if they were collected into groups or classes, and then compared and contrasted to establish relations between them. Observational limitations can then be used to test the validity of whole classes.

A sample of groupings of cosmologies might include:

(i) Time-varying vacuum energy density (e.g., Pollock, 1980; Fabbri, 1980; Salucci and Fabbri, 1983; Freese *et al.*, 1987).

(ii) Background temperature determined by the cosmological constant or function [such as by Gasperini (1987*a,b*, 1988)].

(iii) Attempts at classical or semiclassical cancellations of the cosmological constant (e.g., Hawking, 1984; Abbott, 1985; Ozer and Taha, 1986; Ford, 1982; Reuter and Wetterich, 1987; Barr, 1987; Suen and Will, 1988).

(iv) Observational tests of the notion of a time-varying cosmological function (e.g., Peebles, 1984; Peebles and Ratra, 1988; Ratra and Peebles, 1988; Olson and Jordan, 1987).

(v) Gravitational and cosmological constant calculations from fundamental fields which may be time-varying (e.g., Adler, 1982; Canuto and Lee, 1977; Canuto *et al.*, 1978).

There are many more groupings that probably can be named, but the above gives some indication of the task. There is much to do in this area, but there is much to be gained from resolving these issues.

Thus overall a model-independent investigation of the cosmological function during the inflationary period that consistently arises from the gravitational field equations to the present-day value will itself consistently support the existence of the cosmological "constant."

3. COSMOLOGICAL FUNCTION: A HEURISTIC APPROACH

Our ansatz for the cosmological function consists of the product of two functions. The first function $F(\alpha)$ depends on a period of exponential growth described by the production rate α of the X particle in the early universe during the first phase of inflation. (Note again that we describe our model in terms of a *particle* because of its simplicity of thought *and* because the concept of a "particle" is so complex.) The second function $K(\beta)$ depends directly on the rate of decay β of X. During the first regime (production of X) $F(\alpha)$ dominates and eventually saturates at T_c when production and decay reach equilibrium. During the second regime, decay dominates. Thus we arrive at the following ansatz for the cosmological function

$$\lambda(t) \propto F(\alpha)K(\beta) \tag{1}$$

We will use the above ansatz for the cosmological function to study the behavior of the cosmological scale factor $R(\lambda)$ parametrized with respect to the cosmological function during the inflationary period.

4. EINSTEIN FIELD EQUATIONS

We derive the field equations in the case where the scale factor R is given as a function of the cosmological function λ. Now, during inflation, the scale factor will be parametrized by a time-dependent cosmological function $\lambda(x)$, where x represents a space-time coordinate. By this way, the Robertson–Walker metric (Misner *et al.*, 1973) can be written in the form

$$ds^2 = -dt^2 + R^2(\lambda)[dr^2/(1-kr^2) + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2] \qquad (2)$$

where $R(\lambda)$ will then join smoothly to the usual scale factor $R(t)$ at the end of the inflationary period. It is important to point out that in the Robertson–Walker cosmological metric models, since the scale factor R is already a function of x, then assuming that both R and λ are spatially homogeneous does not destroy the general covariance of the theory (field equations) and therefore does not introduce any preferred time upon the usual choice of comoving coordinates. In what develops, we show that the field equations are perfectly covariant.

In this model, we use the energy-momentum tensor for a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \qquad (3)$$

where

$$T_{00} = \rho(\lambda)$$
$$T_{ij} = {}^3g_{ij}p(\lambda) \qquad (4)$$

We also find it convenient to parametrize both the pressure p and the density ρ as functions of λ. Here the four-velocity u^μ is given in terms of comoving coordinates, and $u^\mu u_\mu = -1$. Using the Einstein field equation

$$G_{\mu\nu} + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad (5)$$

we find the *time-time component*:

$$\dot{\lambda}^2(R'/R)^2 - \lambda/3 = 8\pi G\rho(\lambda)/3 \qquad (6)$$

and the *space-space component*:

$$\ddot{\lambda}(R'/R) + \dot{\lambda}^2(R''/R) + \tfrac{1}{2}\dot{\lambda}^2(R'/R)^2 - \tfrac{1}{2}\lambda = -4\pi Gp(\lambda) \qquad (7)$$

where the prime indicates derivatives with respect to the parameter λ, and the overdot represents the directional derivative along fluid flow lines, $\dot{\lambda} = u^\alpha\partial_\alpha\lambda$, which becomes an ordinary time derivative in the comoving frame.

In the next section, we prove the consistency of the field equations when one has nonzero divergence of the energy-momentum

$$T^\mu{}_{\nu;\mu} \neq 0 \qquad (8)$$

Such models are similar to Rastall's (1972) gravitational theory. It is interesting that a variational approach to such models is very difficult to find (Smalley, 1984).

5. CONSISTENCY OF FIELD EQUATIONS PLUS ENERGY CONSERVATION

The general consistency of the field equations and energy-conservation follows from the Bianchi identity, $G^{\mu}{}_{\nu;\mu}=0$. Applying this to equation (5), we obtain

$$T^{\mu}{}_{\nu;\mu}=\lambda_{,\nu}/(8\pi G) \tag{9}$$

Using equation (3) for the perfect fluid, we obtain

$$\lambda_{,\nu}/(8\pi G)=p_{,\nu}+(-g)^{-1/2}\partial_{\mu}[(-g)^{1/2}(\rho+p)u^{\mu}u_{\nu}]-\Gamma^{s}_{\mu\nu}(\rho+p)u^{\mu}u_{s} \tag{10}$$

Multiplying equation (10) by u^{ν}, recalling that both ρ and p are parametrized functions of λ, and using equation (2), we get the final equation for the conservation of energy

$$\rho'+3(\rho+p)R'/R+(8\pi G)^{-1}=0 \tag{11}$$

which expresses the conservation of energy-momentum as a consequence of equation (8).

Now if one begins with equation (6), takes the directional derivative along the fluid flow lines, and substitutes ρ' from equation (11), one obtains equation (7), which then proves the covariance of this model.

In the next section, we show how to fix the parameters for the cosmological function in our model.

6. COSMOLOGICAL FUNCTION

Based upon our ansatz (1), we write a spatially uniform cosmological function with time dependence as

$$\lambda(t)=A(1-e^{-\alpha t})e^{-\beta t} \tag{12}$$

where A is a constant of proportionality. The parameters A, α, and β are determined from the boundary conditions imposed on $\lambda(t)$. An alternate form for equation (12) can be given by

$$\lambda(t)=A(1-e^{-\alpha t})e^{-\beta t}+\lambda_{p} \tag{13}$$

where λ_p is the value of the cosmological function at the Planck time. We have used both models in our calculations, but both models give the same results. We choose, however, to work with equation (12) for reasons that will become clear by the end of this section. There is, nevertheless, a superficial difference between them in the way one chooses the boundary conditions for $\lambda(t)$ at the Planck time t_p. For example, in (12)

$$\lambda(t_p) = \text{const} \tag{14}$$

In this case, we find that the initial value of λ is a very large positive number; but in using (13) at $t = t_p$, we set

$$\lambda(t_p) = 0 \tag{15}$$

In this case, $\lambda_p \ll 0$ such that $-\lambda_p$ is equal in magnitude to the constant determined in equation (14) for the first model. In using (13), we needed the *additional* boundary condition, $\lambda = 0$ at $t = t_p$. This observation, however, is enough to establish that there is a large, negative Planckian cosmological constant. Since both models seem to give the same results in our calculation, we use the first model for the sake of simplicity in our discussions.

The boundary conditions that we use in our model, equation (12), are as follows: $\lambda(t)$ takes the present-day value of the cosmological constant at the end of the inflationary period

$$\lambda(t_1) = \lambda_0 \tag{16}$$

where $t_1 = 4.8 \times 10^{-38}$ sec is the time at the end of the inflationary period and the beginning of the radiation era (Albrecht *et al.*, 1982; Brandenberger, 1985). The second condition requires that the inflation of the universe be sufficiently large to explain the horizon problem (Guth, 1981; Linde, 1983), so that the change in the scale factor is given by

$$\Delta R \simeq \exp(\hat{\lambda}\tau) \geqq \exp(65) \tag{17}$$

where $\hat{\lambda}$ is the average value of the cosmological function during the inflationary period $\tau = t_1 - t_p$. The third condition involves the temperature T_{RH} at the beginning of the so-called reheat period discussed in some inflationary models. Using the upper and lower limits imposed on T_{RH} by these models (Albrecht *et al.*, 1982; Abbott and Deser, 1982; Krauss, 1983; Linde, 1982; Turner, 1986; Mijie *et al.*, 1986), we have

$$3 \times 10^{16}\ \mathrm{GeV} \geqq T_{\mathrm{RH}} \geqq 10^{14}\mathrm{GeV} \tag{18}$$

or

$$0.217 \times 10^{-40}\ \mathrm{sec} \leqq t' \leqq 0.658 \times 10^{-38}\ \mathrm{sec} \tag{19}$$

where t' corresponds to the beginning of the reheat period. Thus the reheat period is

$$\Delta t' = \sigma^{-1} = t_1 - t' \tag{20}$$

where σ is the unification scale, which is defined in these models (Albrecht *et al.*, 1982)[4] to be in the range 10^{14}–10^{15} GeV. We then apply these boundary conditions to set the parameters in our empirical model.

The first condition, λ_0, is calculated directly from the field equation (5) using the present-day value of the Hubble constant (Kolb and Turner, 1988),

$$H_0 = \dot{R}(t)/R(t) = \dot{\lambda}R'/R = 50 \pm 7 \text{ km sec}^{-1} \text{ Mpc}^{-1} \tag{21}$$

and energy-density, $\rho_0 = 2 \times 10^{-31}$ g/cm^2; then

$$\lambda_0 = 1.04 \times 10^{-35} \text{ sec}^{-2} = 1.155 \times 10^{-56} \text{ cm}^{-2} \tag{22}$$

By comparison, in inflationary models (Albrecht *et al.*, 1982; Brandenberger, 1985; Hawking, 1984),[5] $\lambda_0 \simeq 10^{-82}$ GeV$^2 = 2.25 \times 10^{-34}$ sec^{-2} = 2.68×10^{-55} cm^{-2}, and in the Lemaître (1931, 1933) model $\lambda_0 = 1.4 \times 10^{-35}$ sec$^{-2} = 1.5 \times 10^{-56}$ cm^{-2}.

From the second condition given by (17), we find that in order to have sufficient inflation, then

$$\hat{\lambda}\tau = 65 \tag{23}$$

This means that

$$\int_{t=t_p}^{t=t_1} A(1-e^{-\alpha t})e^{-\beta t}\, dt = \hat{\lambda}\tau = 65 \tag{24}$$

The third condition is imposed by noting that the break between exponential growth and decay occurs at the time when $\dot{\lambda} = 0$. Thus we set this condition so that at $t = t'$

$$\dot{\lambda}(t') = 0 \tag{25}$$

Using the above boundary conditions, we use the Newton–Raphson method (Press *et al.*, 1986) to compute A, a, and β. We find the consistent set of values

$$\begin{aligned} t' &= 0.264 \times 10^{-39} \text{ sec} \quad (\simeq T_{\text{RH}} = 2.49 \times 10^{15} \text{ GeV}) \\ A &= 5.227 \times 10^{44} \text{ sec}^{-2} \\ \alpha &= 1.78 \times 10^{36} \text{ sec}^{-1} \\ \beta &= 3.78 \times 10^{39} \text{ sec}^{-1} \end{aligned} \tag{26}$$

Although we used double-precision arithmetic, we find that the value of t'

[4]See Brandenberger (1985) for a summary of all inflationary models through 1985.
[5]Unit conversions in this paper follow the Appendix of Fliche and Souriau (1992).

given above is not accurate enough to describe the details of inflation near $t = t'$, where $\dot{\lambda}(t') = 0$. This problem also occurs when we use the alternative model given by equation (13). Figure 1, which gives a log-log plot of λ versus t, immediately shows the calculational difficulties due to the large dynamic range of λ, which varies over 75 orders of magnitude from $10^{40}\ \text{sec}^{-2}$ at t' to $\lambda_0 = 1.04 \times 10^{-35}\ \text{sec}^{-2}$ at the beginning of the radiation era at t_1. The value $t' = 0.264 \times 10^{-39}$ sec, which would correspond in this model to $T_{\text{RH}} = 2.49 \times 10^{15}$ GeV, lies within the extreme range given by equations (18)–(19); however, it corresponds to a unification scale that is about one order of magnitude larger than the scale (somewhat) imposed by the *standard* inflationary models with $T_{\text{RH}} \simeq 10^{14}$ GeV (de Vaucouleurs, 1983*b*). Consequently the reheat period, $\Delta t' = 4.77 \times 10^{-38}$ sec ($\simeq 0.137 \times 10^{14}$ Gev) in this model is very fast.

Now we return to the alternative model given by equation (13). We have in this case the same parameters A, α, and β plus the additional parameter λ_p. We use the same boundary conditions, (16)–(17), and (25), with the additional condition given by equation (15). Because of the calculational difficulties near $t = t'$ (due to the discontinuity around that point), we develop a technique which greatly suppresses the magnitude of the discontinuity around that point. Any number can be written as a sum of a series of numbers such that for any product

$$\alpha t' = \sum_{j=1}^{m} \alpha_j \sum_{i=1}^{n} t'_i \tag{27}$$

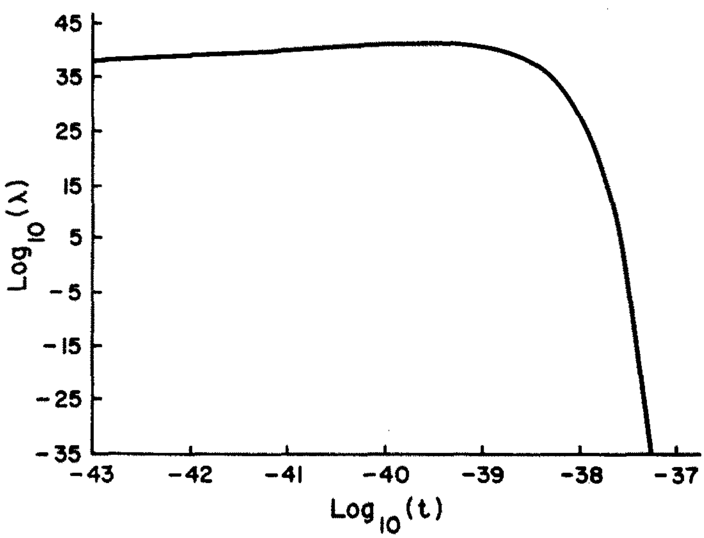

Fig. 1. The cosmological function λ as a function of time from t_p to t_1.

In the calculation we are able to keep terms down to the order of 10^{-73} and use the bisection method (Press *et al.*, 1986) with a tolerance of 10^{-65}. The result gives the consistent set of values

$$\begin{aligned} \alpha &= 6.50 \times 10^{8} \qquad \text{(given)} \\ A &= 6.99 \cdots \times 10^{69} \\ \beta &= 2.643321 \cdots \times 10^{38} \\ \lambda_p &= -6.73200 \cdots \times 10^{35} \\ t' &= 3.78 \cdots \times 10^{-39} \end{aligned} \tag{28}$$

where α is set at the value given and the calculation is continued until we reach a numerical accuracy to the order of 10^{-73}. The dots refer to numbers carried out to nearly 70 decimal places. It is clear from the value of λ_p that in order to eventually connect up with the exact value of $\lambda_0 \simeq 10^{-35}\ \text{sec}^{-2}$, we need to carry out the calculation to at least 70 decimal places, which seems to be beyond the floating-point capability of the CRAY used in this work. As a side result, the value of $t' = 1.76 \times 10^{14}$ GeV corresponds to a reheat period $\sigma^{-1} = 4.42 \times 10^{-38}$ sec ($= 1.49 \times 10^{13}$ GeV), which is very close to the inflationary models discussed above.

For convenience, in the next section, we apply the model given by equation (12) to the field equations (6)–(7) in order to find the scale factor $R(\lambda)$. However, in this model, $\lambda(t_p)$ is very large [see Figure 1 which is obtained from equation (26)] in order to compensate for the appearance of a universe at the beginning of the Big Bang with a very large, negative cosmological constant λ_p.

7. THE COSMIC SCALE FACTOR

In order to determine the cosmic scale factor $R(\lambda)$, we combine the field equation (6) with the hypothesis of the equation of state for stiff matter (Wainwright *et al.*, 1979)

$$\rho(\lambda) = p(\lambda) \tag{29}$$

since in the early universe, the speed of sound in the ultrarelativistic fluid will be comparable with the speed of light. This gives

$$\ddot{\lambda}(R'/R) + \dot{\lambda}^2(R''/R) + 2\dot{\lambda}^2(R'/R)^2 - \lambda = 0 \tag{30}$$

which is a second-order, nonlinear differential equation for $R(\lambda)$ in terms of the functions λ, $\dot{\lambda}$, and $\ddot{\lambda}$, which are known. The function λ is shown in Figure 1, and both functions $\dot{\lambda}$ and $\ddot{\lambda}$ can be obtained by direct differentiation of equation (12) with respect to time; thus

$$\dot{\lambda}(t) = Ae^{-\beta t}[\alpha e^{-\alpha t} - \beta(1 - e^{-\alpha t})] \tag{31}$$

From the numerical calculation, we notice that $\dot{\lambda}(t)$ is a smoothly decreasing function, decreasing from $\dot{\lambda}(t_p) = 9.2 \times 10^{80}$ to $\dot{\lambda}(t') = 0$, and then decreasing further to -10^{79} shortly after t'. Thereafter $\dot{\lambda}$ begins to slowly increase until the end of the reheat period to $\dot{\lambda}(t_1) = -4.47 \times 10^4$. Similarly, $\ddot{\lambda}(t)$ is given by

$$\ddot{\lambda}(t) = Ae^{-\beta t}[\beta^2(1 - e^{-\alpha t}) - 2\alpha\beta e^{-\alpha t} - \alpha^2 e^{-\alpha t}] \tag{32}$$

The function $\ddot{\lambda}$ is an increasing function just until shortly past the inflection point at t', where there is an abrupt change from -3.76×10^{117} to 6.94×10^{118} in a very short time; it then starts to decrease thereafter to the value $\ddot{\lambda}(t_1) = 1.68 \times 10^{44}$ at the end of inflation.

In order to solve equation (30) for the scale factor, we use a backward finite difference method (von Rosenberg, 1969) and find it convenient to define the quantity $f(\lambda)$ such that

$$f(\lambda) = R'/R = \partial(\ln R)/\partial\lambda \tag{33}$$

which reduces equation (30) to the following simplified equation:

$$\dot{\lambda}^2 f' + 3\dot{\lambda} f^2 + \ddot{\lambda} f - \lambda = 0 \tag{34}$$

Note that due to the relative simplicity of equation (34) compared with equation (30), the parametrization of the inflationary period in terms of λ is not only interesting, but useful. In addition, the function $f(\lambda)$ has *well-defined* boundary conditions at the times t' ($\simeq T_{\mathrm{RH}}$) and $t = t_1$ at the beginning of the radiation era. Now during the radiation era, we have

$$R(t) = R_0 t^{1/2} \tag{35}$$

so that

$$f(t_1) = R'/R = \dot{R}/(R\dot{\lambda}) = (2t_1\dot{\lambda})^{-1} = -2.34 \times 10^{32} \text{ sec}^2 \tag{36}$$

At $t = t'$, $\dot{\lambda} = 0$, and then equation (34) gives

$$f(t') = \lambda/\ddot{\lambda} = -8.32 \times 10^{-80} \text{ sec}^2 \tag{37}$$

We depict the variation of $f(\lambda)$ with respect to λ in Figure 2, which is not to scale. However, from this pictorial representation, we can see that $f(\lambda)$ is a decreasing function from $t = t_p$ to $t = t'$, and antisymmetric function about $t = t'$, and continues to decrease until $t = t_1$. The bump near $t = t_1$ comes from trying to match the boundary conditions for the exponential function $\lambda(t)$ to a constant λ_0 at the end of the so-called reheat period at the beginning of the radiation era. We believe that the bump would disappear if we could extend our model calculation to the present epoch instead of matching $\lambda(t)$ at the beginning of the radiation era, i.e., *treating even the present-day* λ_0 *as the long-tail value of the exponential function.* This is an

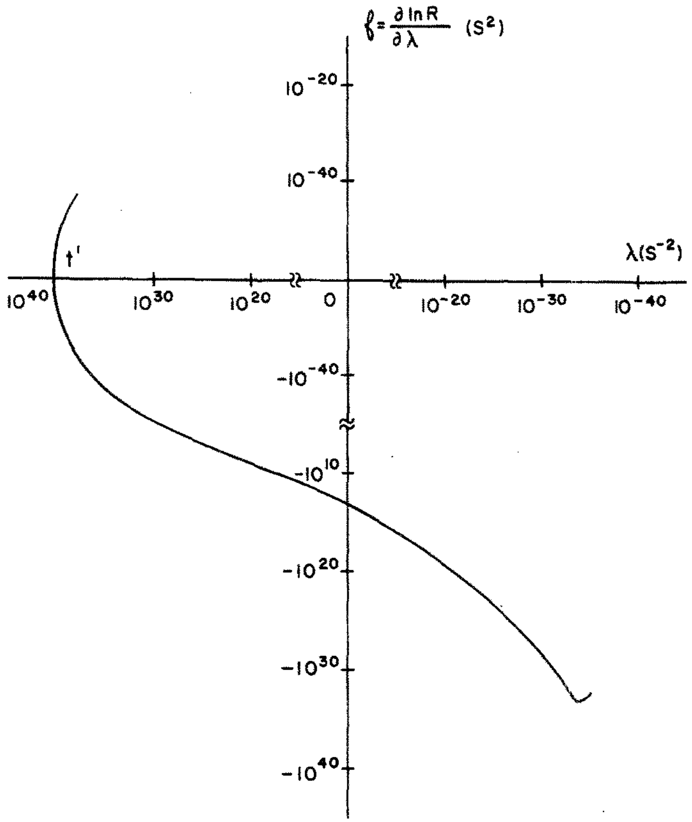

Fig. 2. The function f as a function of λ. Note the broken scales for f and λ.

interesting supposition, but it would complicate our simple treatment of the boundary conditions at $t = t_1$. We will, however, discuss this prospect in our conclusion.

The cosmic scale factor $R(\lambda)$ is obtained by direct integration of $f(\lambda)$. We note that $R(\lambda)$ is a multivalued function of λ. However, this representation has a rather interesting form, as is seen in Figure 3, where we show R/R_0 versus the cosmological function λ. Phase I represents a regime of exponential growth for the cosmological function. During this period, the function R/R_0 increases rapidly (exponentially) until it reaches a steady state where the rate of growth is equal to the rate of decay at the critical temperature T_c. This is followed by a second phase of exponential decay when the effective temperature of the vacuum energy density falls below the threshold energy for growth of $\lambda(x)$. This rapid decrease of λ then

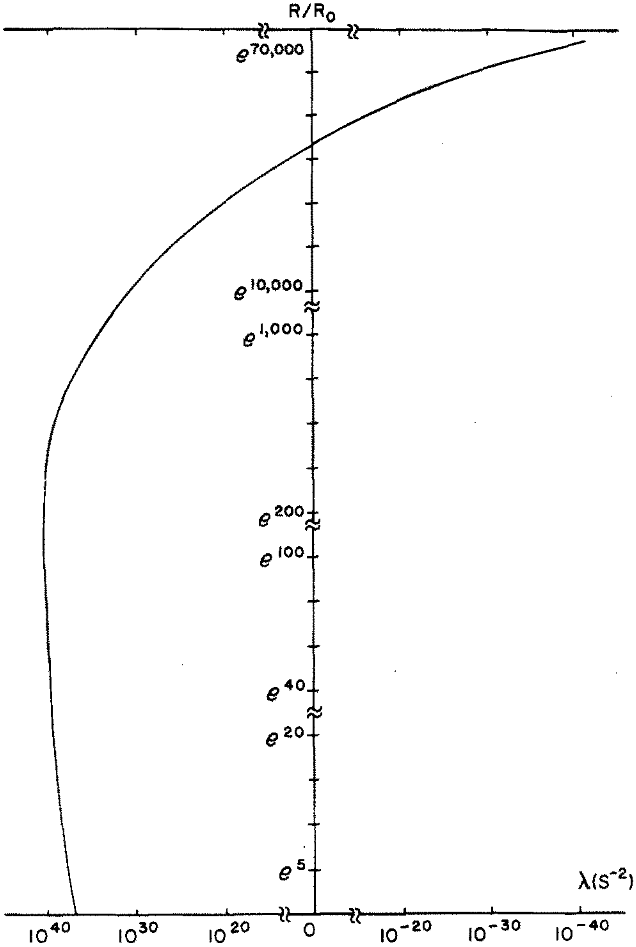

Fig. 3. The cosmological scale factor R as a function of λ. Note the broken scale for R.

represents the end of inflation, i.e., the relaxation of the spacetime curvature so that the scale factor is described by the ordinary Friedmann–Robertson–Walker model (Weinberg, 1972). In Figure 4 we pictorially represent $R(t)/R_0$ itself by breaking the axes in several places. In phase I, from $t = t_p$ to $t = t'$, the exponential growth phase of R/R_0 is very clear. In phase II, it is easy to see the so-called "rollover" of R/R_0 so that it joins the beginning of the radiation era at $t = t_1$. Around $t = t'$, the rapid change from exponential growth to exponential decay in $\lambda(t)$ is very evident at the top of the graph. In the next section, we summarize our results.

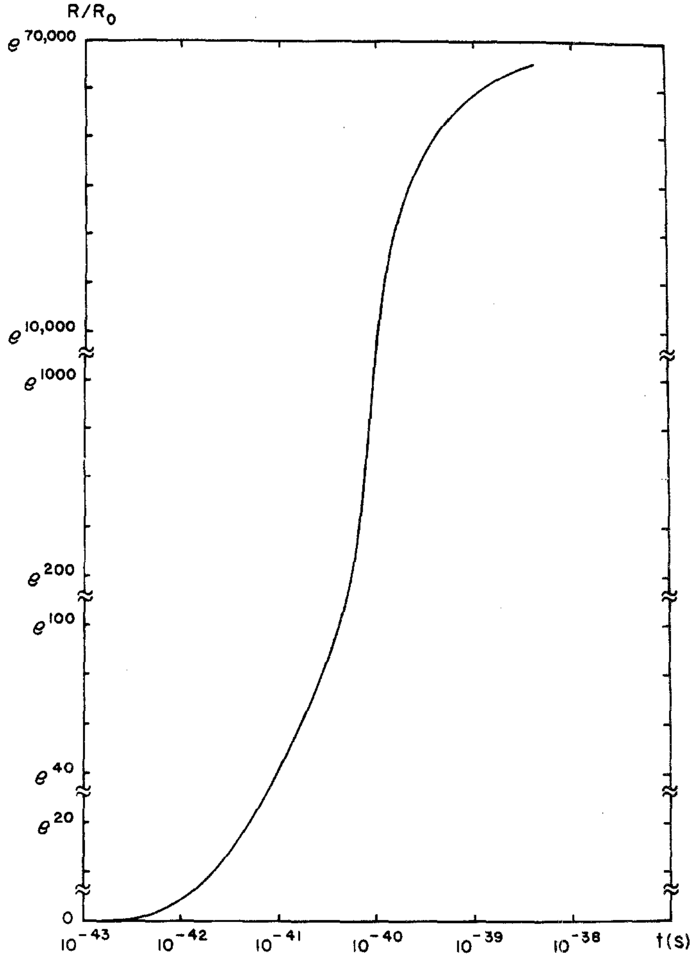

Fig. 4. The cosmological scale factor R as a function of t. Note the broken scale for R.

8. CONCLUSIONS

We have shown how one can model the cosmological function during the inflationary period in order to follow the relatively long period of exponential expansion of the early universe. Such a model is consistently constrained to overcome the problems in the standard Friedmann cosmology (Guth, 1981), for example, horizon and flatness. In the GUT picture of inflation, this exponential expansion of the universe is driven by the false vacuum energy density of the Higgs field which acts like an effective cosmological constant in the Einstein equations. Alternatively, many different underlying particle theories have been proposed. The most popular of these is the Coleman and Weinberg (1973) model. However, none of these theories is without problems, such as very fine-tuning of initial conditions (i.e., fields), or violently fluctuating fields at high temperatures (Mazenko *et al.*, 1985). What we have assumed is that the idea of inflation is an attractive solution to the problems of the standard Big Bang cosmology. However, for all inflationary models, we know that there will be a growth phase which we associate with the cosmological function of the underlying gravitational theory (whether it is due to GUT or to some other quantum or classical theory) and a rapid decay period (whether it is due to super symmetry breaking in GUT or to the fast decay of unstable particles) down to the infinitesimally small cosmological constant ($|\lambda_0| \leqq 10^{-82}$ GeV^2) at the present epoch. There are many dynamical processes (Abbott and Deser, 1982; Starobinsky, 1980; Myhrvold, 1983; Ford, 1985, 1987; Abbott, 1985; Barr, 1987; Reuter and Wetterich, 1987; Peebles and Ratra, 1988) which could dampen a cosmological constant, but our empirical model represents a minimal attempt to address these problems yet provide an overall explanation of what happens: there is an exponential growth regime, followed by a very short, exponential decay period. The exponential growth phase leads in a natural way to the expansion of the universe (inflation), and the second regime leads naturally to the decrease in λ to its present-day value. The change between these two regimes is not overly abrupt as in the GUT model or the standard inflationary models. But on the other hand, the decay period is rather fast (Albrecht *et al.*, 1982; Brandenberger, 1985) compared with the long exponential expansion era, which can be seen from the flat part in Figure 1 for $\log_{10}(\lambda)$ versus $\log_{10}(t)$.

In satisfying the boundary conditions for our ansatz of the cosmological function during the inflationary period, we have discovered what appears to be a very large, negative Planckian cosmological constant λ_p which also must be overcome by any inflationary model. This intrinsic λ_p in effect says that the vacuum is extraordinarily stable, and that in order to have inflation, there must have been an enormous instability/fluctuation in the vacuum itself. However, with our ansatz for $\lambda(t)$, we were able to study the growth of the cosmological scale factor $R(\lambda)$ and eventually $R(t)$ during the inflationary period. This was done by the introduction of the well-defined function $f(\lambda) = R'/R$, which not only simplifies our calculation, but shows that the parametrization of the scale factor with respect to λ is both interesting and useful.

Although the model presented is equivalent to a single, thermodynamic, phase change, it exhibits all the required features of an inflationary period, such as exponential growth of the scale factor plus a natural relaxation of $\lambda(t)$ to the present-day cosmological constant. Since experimentally this constant is known today at best as an upper limit, we speculate that a definitive measurement of the contemporary cosmological constant would provide direct information of the "decay" rate (and therefore a window into the physical process occurring during the inflationary epoch).

We mention an alternative approach to the empirical inflationary model for λ used in this work. If one could find the field equations directly from a Lagrangian density for the vacuum gravitational field, a perfect fluid density, but now containing an additional term which yields a time-dependent cosmological term, the Lagrangian base of such a model would provide an *inherent* self-consistency of the field equations and the conservation laws. Along this line, recent work of Smalley (1993) extends the variational principle for general relativity by extending Riemannian spacetime to Weyl spacetime plus the addition of a surface term in the thermodynamics for the internal energy function in an extended Ray–Einstein–Hilbert variational principle (Ray, 1972) and obtains a time-dependent cosmological function. We are presently looking for solutions for such a spacetime for comparison with this work.

REFERENCES

Abbott, L. F. (1985). *Physics Letters B*, **150**, 427.

Abbott, L. F. (1988). *Scientific American*, **258**, 106.

Abbott, L. F., and Deser, S. (1982). *Nuclear Physics B*, **195**, 76.

Adler, S. L. (1982). *Reviews of Modern Physics*, **54**, 729.

Albrecht, A., Steinhardt, P. J., Turner, M. S., and Wilczek, F. (1982). *Physical Review Letters*, **48**, 1437.

Barr, S. M. (1987). *Physical Review D*, **36**, 1691.

Belinskii, V. A., and Khalatnikov, S. M. (1976). *Soviet Physics JETP*, **42**, 205.

Belinskii, V. A., and Khalatnikov, S. M. (1977). *Soviet Physics JETP*, **45**, 1.

Blome, H. J., and Priester, W. (1985). *Astrophysics and Space Science*, **112**, 323.

Brandenberger, R. (1985). *Reviews of Modern Physics*, **57**, 1.

Canuto, V., and Lee, J. F. (1977). *Physics Letters B*, **72**, 281.

Canuto, V., Hsieh, S. H., and Adams, P. F. (1978). *Physical Review D*, **18**, 3577.

Coleman, S., and Weinberg, E. J. (1973). *Physical Review D*, **7**, 1988.

Dabrowski, M., and Stelmach, J. A. (1982). *Astronomical Journal*, **94**, 1373.

Dabrowski, M., and Stelmach, J. A. (1986). *Astronomical Journal*, **92**, 1272.

De Vaucouleurs, G. (1982). *Nature*, **299**, 303.

De Vaucouleurs, G. (1983*a*). *Astrophysical Journal*, **268**, 451.

De Vaucouleurs, G. (1983*b*). *Astrophysical Journal*, **268**, 468.

Fabbri, R. (1979). *Lettere al Nuovo Cimento*, **26**, 77.

Fabbri, R. (1980). *Nuovo Cimento*, **56B**, 125.

Fliche, H. H., and Souriau, J. M. (1992). *Astronomy and Astrophysics*, **78**, 87.

Ford, L. H. (1982). *Physical Review D*, **35**, 2339.

Ford, L. H. (1985). *Physical Review D*, **31**, 710.
Ford, L. H. (1987). *Physical Review D*, **35**, 2339.
Freese, K., Adams, F. C., Frieman, J. A., and Mottola, E. (1987). *Nuclear Physics B*, **287**, 797.
Gasperini, M. (1987*a*). *Physics Letters B*, **194**, 347.
Gasperini, M. (1987*b*). *Astrophysics and Space Science*, **138**, 387.
Gasperini, M. (1988). *Classical and Quantum Gravity*, **5**, 521.
Georgi, H., and Glashow, S. L. (1974). *Physical Review Letters*, **32**, 438.
Guth, A. H. (1981). *Physical Review D*, **23**, 347.
Hawking, S. W. (1984). *Physics Letters B*, **134**, 403.
Hoell Chu, Y., Blome, H. J., and Priester, W. (1988). *Astrophysics and Space Science*, **148**, 119.
Jensen, L. G., and Stein-Schabes, J. A. (1986). *Physical Review D*, **34**, 931.
Jensen, L. G., and Stein-Schabes, J. A. (1987). *Physical Review D*, **35**, 1146.
Klapdor, H. V., and Grotz, K. (1986). *Astrophysical Journal*, **301**, L39.
Kolb, E. W., and Turner, M. S. (1988). *The Early Universe*, Addison-Wesley, New York.
Krauss, L. L., (1983). *Nuclear Physics B*, **227**, 556.
Lemaître, G. (1931). *Monthly Notices of the Royal Astronomical Society*, **91**, 483.
Lemaître, G. (1933). *Academie des Sciences Paris Comptes Rendus*, **196**, 903, 1085.
Linde, A. D. (1982). *Physics Letters B*, **108**, 389.
Linde, A. D. (1983). *Physics Letters B*, **129**, 177.
Loh, E. D. (1986). *Physical Review Letters*, **57**, 2865.
Loh, D. E. (1988). *Astrophysical Journal*, **329**, 24.
Mazenko, G. F., Unruh, W. G., and Wald, R. M. (1985). *Physical Review D*, **31**, 273.
Mijie, M. B., Morris, M. S., and Suen, W. M. (1986). *Physical Review D*, **34**, 2934.
Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*, Freeman, New York.
Myhrvold, N. D. (1983). *Physics Letters B*, **132**, 132.
Nikomarov, E. S., and Khalatnikov, S. M. (1978). *Soviet Physics JETP*, **48**, 492.
Olson, T. S., and Jordan, T. F. (1987). *Physical Review D*, **35**, 3258.
Ozer, M., and Taha, M. O. (1986). *Physics Letters B*, **171**, 363.
Peebles, P. J. E. (1984). *Astrophysical Journal*, **284**, 439.
Peebles, P. J. E., and Ratra, B. (1988). *Astrophysical Journal*, **325**, L17.
Pollock, M. D. (1980). *Monthly Notices of the Royal Astronomical Society*, **193**, 825.
Press, W. H., Flannery, B. P., Teukolsky, S. A., and Wetterling, W. T. (1986). *Numerical Recipes: The Art of Scientific Computing*, Cambridge University Press, Cambridge.
Rastall, P. (1972). *Physical Review D*, **6**, 3357.
Ratra, B., and Peebles, P. J. E. (1988). *Physical Review D*, **37**, 3406.
Ray, J. R. (1972). *Journal of Mathematical Physics*, **13**, 1451.
Reuter, H., and Wetterich, C. (1987) *Physics Letters B*, **188**, 38.
Reuter, M., and Wetterich, C. (1987). *Physics Letters B*, **188**, 38.
Salucci, R., and Fabbri, R. (1983). *Nuovo Cimento*, **77B**, 62.
Smalley, L. L. (1984). *Nuovo Cimento*, **80**, 42.
Smalley, L. L. (1993). *Classical and Quantum Gravity*, **10**, 1179.
Starobinsky, A. A. (1980). *Physics Letters B*, **91**, 99.
Suen, W. M., and Clifford, M. W. (1988). *Physics Letters B*, **205**, 447.
Suen, W. M., and Will, C. M. (1988). *Physics Letters B*, **205**, 442.
Thielmann, T. K., Metzinger, J., and Klapdor, H. V. (1983). *Zeitschrift für Physik*, **309**, 301.
Turner, M. S. (1986). In *Proceedings of 11th Workshop on General Relativity and Gravitation, Stockholm, Sweden*, M. A. H. MacCallum, ed., Queen Mary College, London.
Turner, M. S., and Widrow, L. M. (1986). *Physical Review Letters*, **57**, 2237.

Turner, M. S., Stergman, G., and Krauss, L. M. (1984). *Physical Review Letters*, **52**, 2090.
Von Rosenberg, D. W. (1969). *Methods for the Numerical Solutions of Partial Differential Equations*, Eldrbire, New York.
Wainwright, J., Ince, W. C. C., and Marsh, B. J. (1979). *General Relativity and Gravitation*, **10**, 259.
Wald, R. M. (1983). *Physical Review D*, **28**, 2118.
Weber, E. (1984). *Journal of Mathematical Physics*, **25**, 3279.
Weber, E. (1985). *Journal of Mathematical Physics*, **26**, 1306.
Weber, E. (1987). *Journal of Mathematical Physics*, **28**, 1658.
Weber, E. (1988). *Journal of Mathematical Physics*, **27**, 1578.
Weinberg, S. (1982). *Physical Review Letters*, **59**, 2607.
Weinberg, S. (1972). *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley, New York.
Zeldovitch, T. B. (1968). *Soviet Physics Uspekhi*, **11**, 381.
Zeldovitch, T. B. (1981). *Soviet Physics Uspekhi*, **24**, 216.